Math 113 (Calculus II)
Test 2 Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Integrate $\int \frac{x^3+2x+2}{x^3-4x}\,dx$ using partial fractions. Which of the following **cannot** be in the solution?

   a) $\ln|x-2|$ b) $\ln|x+2|$ c) x

   d) $\ln|x|$ e) $\frac{1}{x}$ f) None of the above.

   Solution: e)

2. Which trigonometric substitution should be used for the integral $\int \sqrt{4x^2-3}\,dx$?

   a) $x = \frac{\sqrt{3}}{2}\sin u$ b) $x = \frac{\sqrt{3}}{2}\tan u$ c) $x = \frac{\sqrt{3}}{2}\sec u$

   d) $x = \frac{3}{4}\sin u$ e) $x = \frac{3}{4}\tan u$ f) $x = \frac{3}{4}\sec u$

   g) $x = \frac{2}{\sqrt{3}}\sin u$ h) None of the above.

   Solution: c)

3. What is the value of the definite integral $\int_0^4 \sqrt{x^2+9}\,dx$?

   a) $2 - \frac{9}{2}\ln 3$ b) $\frac{9}{3}\ln 2$ c) $5 + \frac{9}{2}\ln 5$

   d) $3 - 5\ln 3$ e) $9 - \frac{5}{2}\ln 3$ f) $5\ln 3$

   g) $10 + \frac{9}{2}\ln 3$ h) $\frac{9}{2}\ln 3$ i) None of the above.

   Solution: g)

4. Give the integral definition of $\ln x$.

a) $\int_0^x \frac{1}{t} dt$ for $x > 0$.

b) $\int_1^x \frac{1}{t} dt$ for $x > 1$.

c) $\int_1^x \frac{1}{t} dt$ for all real numbers x.

d) $\int_1^x \frac{1}{t} dt$ for $x > 0$.

e) $\int_1^x \frac{1}{t^2} dt$ for $x > 0$.

f) $\int_1^4 \frac{1}{x} dx$.

g) None of the above.

Solution: d)

5. If $a > 0$ and x is any real number, define a^x.

a) a^x is a times itself x times.

b) $a^x = \ln x$

c) $a^x = e^{x \ln a}$

d) $a^x = e^{\ln(a^x)}$

e) $a^x = e^{\ln(a)}$

f) None of the above.

Solution: c)

6. Find $\int_{-\infty}^{\infty} xe^{-x^2}\, dx$.

a) 0

b) 1

c) -1

d) 2

e) -2

f) e^1

g) diverges

h) None of the above.

Solution: a)

7. Find $\int_{-2}^{2} \frac{1}{x^4}\, dx$.

a) 0

b) $\frac{1}{12}$

c) $\frac{-1}{12}$

d) $\frac{1}{4}$

e) $\frac{-1}{4}$

f) $-\frac{1}{3^4} + \frac{1}{24}$

g) diverges

h) None of the above.

Solution: g)

8. If $f'(x) > 0$ and $f''(x) < 0$ for $a \leq x \leq b$, which of the following is true? L_n is the approximation of the integral using n subdivisions and the left end point, R_n uses the right end point, M_n uses the Midpoint Rule, T_n uses the Trapezoidal Rule.

a) $L_n < M_n < T_n < R_n$ b) $L_n < M_n < R_n < T_n$ c) $R_n < L_n < T_n < M_n$

d) $L_n < R_n < T_n < M_n$ e) $R_n < L_n < M_n < T_n$ f) $R_n < M_n < T_n < L_n$

g) $L_n < T_n < M_n < R_n$ h) None of the above

Solution: g)

Free response: Give your answer in the space provided. Answers not placed in this space will be ignored. 6 points each

9. (8 points) Integrate $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

Solution: Let $u = 1 + x$. Then, $du = dx$, and $x = u - 1$. The integral becomes

$$\int_1^2 \sqrt{\frac{2-u}{u}}\, du = \int_1^2 \sqrt{\frac{2}{u} - 1}\, du.$$

Let $\sqrt{\frac{2}{u}} = \sec\theta$, or $\sqrt{u} = \sqrt{2}\cos\theta$. Then,

$$\frac{1}{2\sqrt{u}}\, du = -\sqrt{2}\sin\theta\, d\theta,$$

and

$$du = -2\sqrt{2}\sqrt{u}\sin\theta\, d\theta = -4\cos\theta\sin\theta\, d\theta.$$

Thus, the integral becomes

$$\int_1^2 \sqrt{\frac{2}{u} - 1}\, du = \int_{\sec^{-1}(\sqrt{2})}^{\sec^{-1}(1)} \sqrt{\sec^2\theta - 1}(-4)\cos\theta\sin\theta\, d\theta$$

$$= 4\int_{\sec^{-1}(1)}^{\sec^{-1}(\sqrt{2})} \tan\theta\cos\theta\sin\theta\, d\theta$$

$$= 4\int_{\sec^{-1}(1)}^{\sec^{-1}(\sqrt{2})} \sin^2\theta\, d\theta$$

$$= 4\int_{\sec^{-1}(1)}^{\sec^{-1}(\sqrt{2})} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= 4\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\Big|_{\sec^{-1}(1)}^{\sec^{-1}(\sqrt{2})} = 4\left(\frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta\right)\Big|_{\sec^{-1}(1)}^{\sec^{-1}(\sqrt{2})}$$

Note that $\sin(\sec^{-1}(a)) = \frac{\sqrt{a^2-1}}{a}$ and $\cos(\sec^{-1}(a)) = \frac{1}{a}$. Hence the above becomes

$$2\sec^{-1}(\sqrt{2}) - 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{\pi}{2} - 1.$$

OR

Let $u = \sqrt{1+x}$. Then, $u^2 = 1 + x$, and $2u\,du = dx$. Also, $x = u^2 - 1$. Notice that

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_1^{\sqrt{2}} \frac{\sqrt{2-u^2}}{u} 2u\,du$$
$$= 2\int_1^{\sqrt{2}} \sqrt{2-u^2}\,du.$$

Let $u = \sqrt{2}\sin\theta$. then $du = \sqrt{2}\cos\theta\,d\theta$, and

$$2\int_1^{\sqrt{2}} \sqrt{2-u^2}\,du = 2\int_{\pi/4}^{\pi/2} \sqrt{2-2\sin^2\theta}\sqrt{2}\cos\theta\,d\theta$$
$$= 4\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta = 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta$$
$$= 4\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)|_{\pi/4}^{\pi/2}$$
$$= 4\left(\frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta\right)|_{\pi/4}^{\pi/2}$$
$$= \pi + 0 - \frac{\pi}{2} - 2 * \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{2} - 1.$$

10. (8 points)Evaluate the indefinite integral $\int \frac{x+4}{x^2+2x+5}\,dx$.

Solution:

$$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx$$

The first integral can be solved by substitution: Let $u = x^2+2x+5$. Then, $du = (2x+2)\,dx$, or $du/2 = (x+1)\,dx$.

$$\int \frac{x+1}{x^2+2x+5}\,dx = \int \frac{du}{2u} = \frac{1}{2}\ln u$$
$$= \frac{1}{2}\ln(x^2+2x+5) + C.$$

To solve the second, we complete the square:

$$\int \frac{3}{x^2+2x+5}\,dx = \int \frac{3}{(x+1)^2+4}\,dx.$$

Letting $x+1 = 2\tan\theta$, we have $dx = 2\sec^2\theta d\theta$. Thus,

$$\int \frac{3}{(x+1)^2+4}\,dx = \int \frac{3}{4\tan^2\theta+4} 2\sec^2\theta d\theta$$
$$= \frac{3}{2}\int d\theta = \frac{3}{2}\theta + C$$
$$\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$$

Thus, the solution is

$$\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\arctan(\frac{x+1}{2}) + C.$$

11. (6 points) Evaluate the indefinite integral $\int \frac{3}{(4-x^2)^{3/2}}\,dx$.

Solution: Let $x = 2\sin u$. Then $dx = 2\cos u\,du$, and

$$\int \frac{3}{(4-x^2)^{3/2}}\,dx = \int \frac{3\cdot 2\cos u}{(4-4\sin^2 u)^{3/2}}\,du$$

$$= \frac{6}{8}\int \frac{\cos u}{\cos^3 u}\,du = \frac{3}{4}\sec^2 u\,du = \frac{3}{4}\tan u + C$$

Since $\sin u = \frac{x}{2}$, then $\tan u = \frac{x}{\sqrt{4-x^2}}$, and the integral is

$$\frac{3x}{4\sqrt{4-x^2}} + C.$$

12. (6 points) Prove the third law of logarithms: $\ln(x^r) = r\ln x$ where $x > 0$ and r is rational.

Solution: Notice that

$$\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}rx^{r-1} = r\frac{1}{x} = r\frac{d}{dx}\ln(x).$$

Thus, $\ln(x^r) = r\ln(x) + C$ for some constant C. If $x = 1$, we see that $\ln(1) = r\ln(1) + C$, or $0 = C$. Thus, $\ln(x^r) = r\ln(x)$.

13. (8 points) Use the comparison theorem to determine if

$$\int_1^\infty \frac{x}{x^4+1}\,dx$$

is convergent or divergent.

Solution: Notice that

$$\frac{x}{x^4+1} < \frac{x}{x^4} = \frac{1}{x^3}.$$

Thus,

$$\int_1^\infty \frac{x}{x^4+1}\,dx < \int_1^\infty \frac{1}{x^3}\,dx.$$

Since the right integral converges, the left one does as well by the comparison theorem.

14. (8 points) Use Simpson's Rule with n=4 to approximate $\int_1^5 \frac{1}{x}\,dx$.

Solution:

$$\frac{1}{3}\left(1 + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}\right)$$

$$= \frac{1}{3}\left(1 + 2 + \frac{2}{3} + 1 + \frac{1}{5}\right) = \frac{1}{3}\left(4 + \frac{13}{15}\right) = \frac{73}{45}.$$

15. (8 points) Integrate $\int \frac{\sin 2x}{1+\sin^2 x}\,dx$

Solution:

$$\int \frac{\sin 2x}{1+\sin^2 x}\,dx = \int \frac{2\sin x\cos x}{1+\sin^2 x}\,dx$$

This can be integrated using two substitutions, or just one. The two substitutions would be to let $u = \sin x$ and $du = \cos x$, and then letting $v = 1 + u^2$, $dv = 2u\,du$. However, we can accomplish both at the same time by setting

$$u = 1 + \sin^2 x\,dx, \quad du = 2\sin x \cos x\,dx.$$

Then,

$$\begin{aligned}\int \frac{2\sin x \cos x}{1+\sin^2 x}\,dx &= \int \frac{du}{u} = \ln|u| + C \\ &= \ln(1+\sin^2 x) + C.\end{aligned}$$

16. (8 points) Find $\displaystyle\int \frac{3x^2+x+6}{(x-1)(x^2+9)}\,dx$.

Solution:

$$\begin{aligned}\frac{3x^2+x+6}{(x-1)(x^2+9)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \\ 3x^2+x+6 &= A(x^2+9) + (Bx+C)(x-1)\end{aligned}$$

If $x = 1$, then the above becomes $10 = 10A$, so $A = 1$. If $x = 0$, then we get $6 = 9A - C = 9 - C$, so $C = 3$. If $x = -1$, then $8 = 10A + 2B - 2C = 10 + 2B - 6$, so $B = 2$.

Thus,

$$\begin{aligned}\int \frac{3x^2+x+6}{(x-1)(x^2+9)}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{2x+3}{x^2+9}\,dx \\ &= \int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+9}\,dx + \frac{3}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx \\ &= \ln|x-1| + \ln(x^2+9) + \arctan(\frac{x}{3}) + C.\end{aligned}$$

END OF EXAM